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Flow value lemma

Proof:

$$v(f) = \sum_{e \text{ out of } s} f(e)$$

$$= \sum_{v \in A} \left(\sum_{e \text{ out of } v} f(e) - \sum_{e \text{ into } v} f(e) \right) \qquad (1)$$

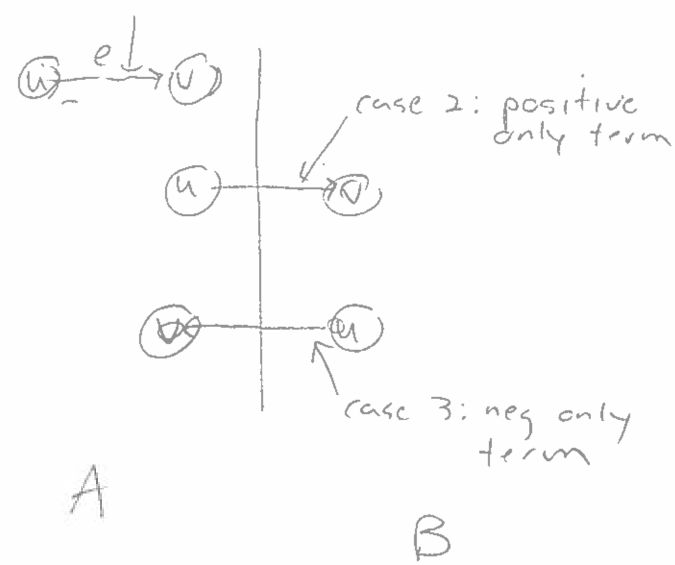
$$= \sum_{e \text{ out of } A} f(e) - \sum_{e \text{ into } A} f(e) \qquad (2)$$

(1) justification

$$\sum_{e \text{ out of } v} f(e) - \sum_{e \text{ into } v} f(e) = \begin{cases} 0 & v \neq s \\ v(f) & v = s \end{cases}$$

(2) justification

case 1: terms cancel



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<u>Corollary:</u> Let f be an s-t flow and let (A,B) be any s-t cut. Then $v(f) \le c(A,B)$.

<u>Proof:</u>

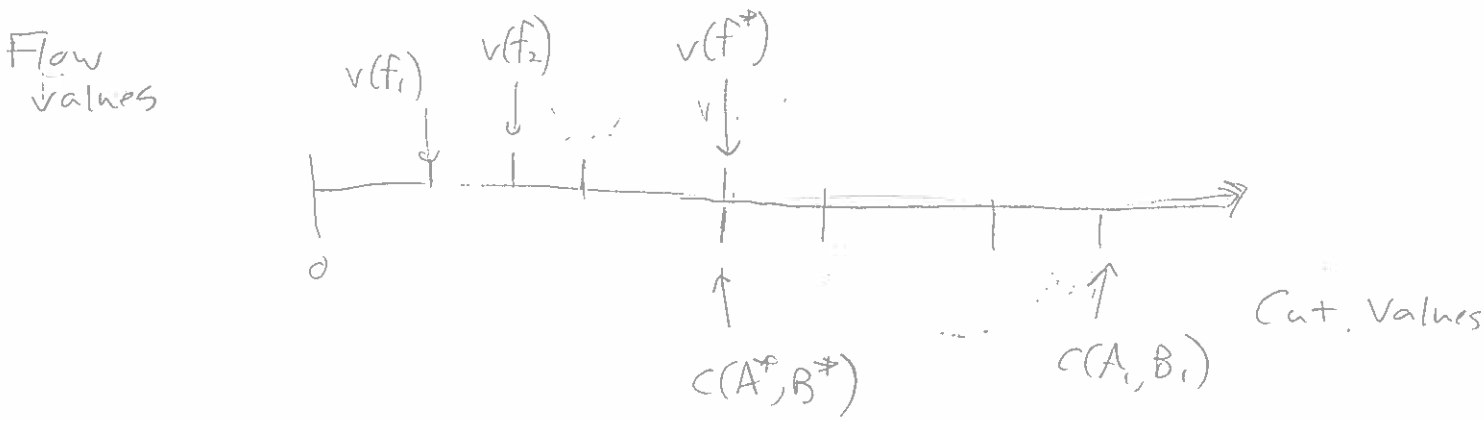
$$v(f) = \sum_{e \text{ out of } A} f(e) - \sum_{e \text{ into } A} f(e)$$

$$\le \sum_{e \text{ out of } A} f(e)$$

$$\le \sum_{e \text{ out of } A} c(e)$$

$$= c(A,B)$$

<u>Illustration:</u>



If $v(f^*) = c(A^*,B^*)$, then $v(f^*) \ge v(f)$ for all f.
and $c(A^*,B^*) \le c(A,B)$ for all (A,B).
$\Rightarrow$ f^* is a max flow <u>and</u> (A^*,B^*) is a min cut

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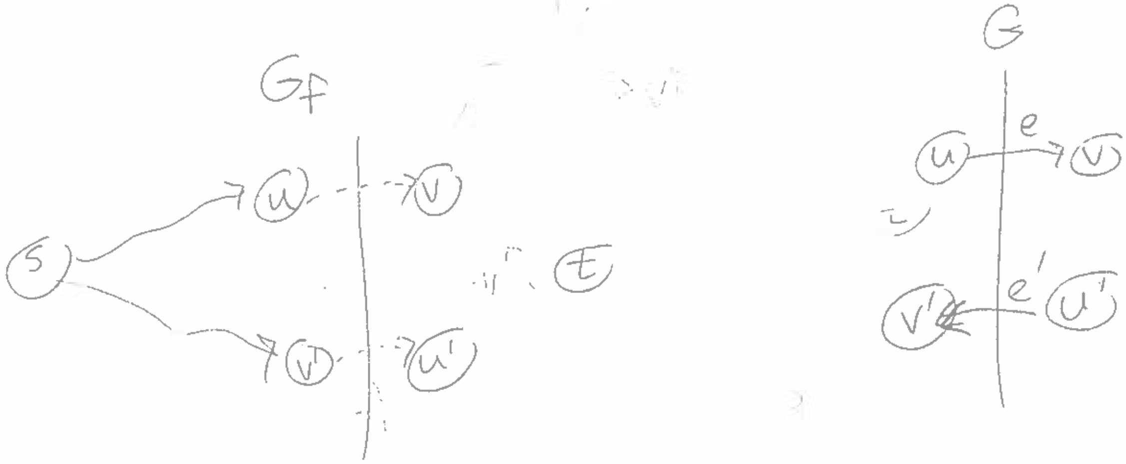
Max-Flow Min-Cut Theorem Proof

<u>Proof</u>: Consider G_f + G



<u>Goal</u>: show that all edges leaving A are saturated, and all edges into A have zero flow, so that

$$v(f) = \sum_{e \text{ out of } A} f(e) - \sum_{e \text{ into } A} f(e)$$

$$= \sum_{e \text{ out of } A} c(e) - \sum_{e \text{ into } A} 0 = c(A,B)$$

Let $e=(u,v)$ be an edge out of A in G. Then $f(e)=c(e)$, for otherwise the forward edge (u,v) is in G_f, so there is a $s \rightsquigarrow v$ path, and $v \in A$. $\Rightarrow \Leftarrow$

Let $e'=(u',v')$ be an edge <u>into</u> A in G. Then $f(e')=0$, otherwise the reverse edge (v',u') is in G_f, so there is an $s \rightsquigarrow u'$ path. $\Rightarrow \Leftarrow$

$\square$